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Infinite-dimensional turbulence

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Abstract. We have investigated infinite Reynolds number homogeneous isotropic turbulence for space dimensions $d \to \infty$ looking for possible simplifications. The calculations were done using both short-time expansions and renormalised expansions. For $d \to \infty$ non-linear interactions become confined to triads of wavevectors having one right angle. To all orders in perturbation the spectrum of the kinetic energy per mass has a finite limit provided a rescaled time $\tilde{t} = t/\sqrt{d}$ is used. It is shown that the incompressibility constraint does not drop out in infinite dimensions. No particular simplification has been found in any class of graphs which would be comparable to what happens in the large-d or high-n limits of critical phenomena.

1. Introduction

Over the last few years there have been several attempts to implement in the statistical theory of turbulence some of the concepts and tools which have been valuable in field theory and the theory of critical phenomena (de Gennes 1975; Martin *et al* 1973; Nelkin 1974, 1975; see Rose and Sulem 1977 for review). In particular there has been a search for 'crossover' dimensions above or below which the calculation becomes trivial. For certain problems involving the largest eddies this search has been successful (Forster *et al* 1976, 1977). There remains however the outstanding problem of three-dimensional fully developed turbulence where deviations to Kolmogorov's 1941 predictions are generally ascribed to the strongly non-Gaussian, intermittent character of the small scales (Frisch *et al* 1977, Kolmogorov 1941 (also 1968), 1962, Kraichnan 1974a). This problem does not seem to simplify in any finite space dimension d, although the properties of the energy cascade are found to be very much dependent on dimension near $d = 2$ (Frisch *et al* 1976, Fournier and Frisch 1977a). It has been suggested that fully developed turbulence simplifies as $d \to \infty$ (A A Migdal 1976 and E D Siggia 1976, private communications). Furthermore it has been shown by Kraichnan (1974b) for the advection of a passive scalar by prescribed velocity fields with large-scale spatial gradients and white-noise time dependence, that the fluctuations in the small-scale distribution of the passive scalar go to zero as $d \to \infty$. It is also known that in critical phenomena taking the limit $d \to \infty$ or $n \to \infty$ (n = number of spin components) results in drastic simplifications and that various quantities can be expanded in powers of $1/d$ or $1/n$ (Gerber and Fischer 1974, 1975, Ma 1976).

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It is the purpose of this paper to start an investigation of the $d \to \infty$ limit for homogeneous isotropic fully developed turbulence. In order to take the limit we have to continue analytically the dimension of space. This can be done as in other fields of theoretical physics (Leibbrandt 1975), term by term in the systematic perturbation expansions of the relevant statistical quantities, such as the energy spectrum.

2. Taylor expansions in time

The Navier–Stokes equation in d (integer) dimensions reads

$$\frac{\partial V_i}{\partial t}+\sum_{j=1}^{d} V_j\frac{\partial V_i}{\partial X_j}=-\frac{\partial p}{\partial X_i}+\nu\nabla^2 V_i \qquad i=1,\ldots,d \tag{2.1}$$

$$\sum_{i=1}^{d}\frac{\partial V_i}{\partial X_i}=0. \tag{2.2}$$

Dropping viscosity (see below for the $\nu \to 0$ limit) and noticing that the pressure is a quadratic functional of the velocity we may rewrite the Navier–Stokes equation symbolically as

$$\dot V = VV.$$

Taking successive time derivatives evaluated at $t=0$ we then obtain a formal Taylor series

$$V(t)=V_0+tV_0V_0+t^2V_0V_0V_0+\ldots \tag{2.3}$$

where V_0 stands for the initial velocity field. If the initial conditions are random, any single, or multiple, time moment of the velocity field can be expressed as a single or multiple time Taylor series involving the initial moments. This takes a particularly simple form when initial conditions are Gaussian homogeneous isotropic of zero mean as will be assumed henceforth. We then obtain for the energy spectrum E (see § 3 for precise definition)

$$E(t)=E_0+t^2E_0E_0+t^4E_0E_0E_0+\ldots. \tag{2.4}$$

Odd-order terms vanish by the Gaussian assumption. As we shall see it is then easy to continue this expansion term by term into arbitrary, possibly non-integer, dimensions.

What do we know about the convergence properties of such expansions and their relation to the actual turbulence problem? Recall that for the infinite Reynolds number problem one must carefully take the limit $\nu \to 0$, which is certainly not the same as putting $\nu=0$ from the start (Brissaud *et al* 1973, Orszag 1976, Rose and Sulem 1977). *A priori* there is no reason to believe that the series (2.4) has more than a zero radius of convergence (Kraichnan 1966, 1970): indeed, individual realisations of the inviscid Navier–Stokes equation (Euler equation) in any dimension $d>2$ are likely to blow up at a finite time which by the Gaussian assumption can be arbitrarily close to $t=0$. This question has recently been investigated on the Burgers equation which is known to produce singularities at a finite time. Fournier and Frisch (1977b) have shown that for any finite wavenumber k this formal Taylor series (in the above sense) of the energy spectrum around $t=0$ has an infinite radius of convergence. There are also strong indications that the formal solution differs from the true solution

($\nu \to 0$) by a non-analytic function with an identically vanishing Taylor series (something like $\exp(-1/t^2)$) and therefore constitutes a very good approximation for short times. The non-analytic part stems from the very small fraction of realisations which have produced a singularity between 0 and t.

For the Navier–Stokes equation the convergence and approximation properties of the formal expansion are unknown. Still, we shall use this expansion to investigate the short-time expansion of the statistical Navier–Stokes equation in d dimensions (§§ 3 and 4).

The analytic continuation in dimension could conceivably be done on other expansions. For a finite positive viscosity it is possible to expand in powers of the non-linear term (Reynolds number expansion). This expansion coincides with the above formal one at zero viscosity; its convergence properties are not known (see Kraichnan 1966, 1970). For the purpose of this paper the Reynolds number expansion is essentially equivalent to the Taylor expansion. Finally there are the so called renormalised expansions (Martin *et al* 1973, Kraichnan 1977 and private communication) the lowest order of which is the direct interaction approximation (DIA: see § 5).

3. Second-order expansion

Much of the structure of the formal expansion is already seen at second order. For statistically homogeneous initial conditions it is convenient to work in Fourier space, in which the Navier–Stokes equation is represented by

$$\frac{\partial V_l(\boldsymbol{k}, t)}{\partial t} + \nu k^2 V_l(\boldsymbol{k}, t) = \frac{-\mathrm{i}}{2} \int_{\boldsymbol{p}+\boldsymbol{q}=\boldsymbol{k}} P_{lmn}(\boldsymbol{k}) V_m(\boldsymbol{p}, t) V_n(\boldsymbol{q}, t)\, \mathrm{d}^d p \tag{3.1}$$

where

$$P_{lmn}(\boldsymbol{k}) = k_m P_{ln}(\boldsymbol{k}) + k_n P_{lm}(\boldsymbol{k})$$

and

$$P_{ln}(\boldsymbol{k}) = \delta_{ln} - \frac{k_l k_n}{k^2}$$

is the transverse projection operator which arises from the pressure. The series (2.4) is now a functional of the velocity covariance

$$\langle V_i(\boldsymbol{k}, t) V_j(\boldsymbol{p}, t')\rangle \equiv U_{ij}(\boldsymbol{k};\, t, t')\delta^{(d)}(\boldsymbol{k}+\boldsymbol{p}) \equiv \frac{P_{ij}(\boldsymbol{k})}{d-1} \frac{2E(k;\, t, t')}{S_d k^{d-1}} \delta^{(d)}(\boldsymbol{k}+\boldsymbol{p}) \tag{3.2}$$

evaluated at the initial time $t = t' = 0$. The following notation has been used. S_d is the surface of the d-dimensional unit sphere:

$$S_d = 2\frac{\pi^{d/2}}{\Gamma(d/2)} \tag{3.3}$$

and the energy spectrum $E(k;\, t, t')$ is related to the mean kinetic energy per mass by

$$\int_0^\infty E(k;\, t, t')\, \mathrm{d}k = \tfrac{1}{2}\langle \boldsymbol{V}^2(t)\rangle. \tag{3.4}$$

When the tensor indices in (2.3) and (2.4) are made explicit, we obtain the formal expansion for U_{ij}, whose trace yields the expansion for E. To second order (setting $\nu=0$):

$$E(k;\, t, t)=E(k;\, 0, 0)+t^2\frac{4S_{d-1}}{(d-1)^2S_d}\int_{\Delta_k} \mathrm{d}p\, \mathrm{d}q \frac{k}{pq} b^{(d)}(k, p, q)$$
$$\times(\sin^{d-3}\alpha\, k^2E(p;\, 0, 0)E(q;\, 0, 0)$$
$$-\sin^{d-3}\beta\, p^2E(q;\, 0, 0)E(k;\, 0, 0))+\mathrm{O}(t^4). \tag{3.5}$$

The coefficient $b^{(d)}$ resulting from various contractions of P_{ij} operators reads

$$b^{(d)}(k, p, q)=\frac{p}{2k}[(d-3)Z+(d-1)XY+2Z^3].$$

α, β and γ are the angles opposite k, p, q in the k, p, q triangle, X, Y and Z being their cosines. The d-dimensional Fourier space integration has been reduced to one over a strip Δ_k (see figure 1) in the p, q plane such that k, p, q can form a triangle (see Rose and Sulem 1977 for details).

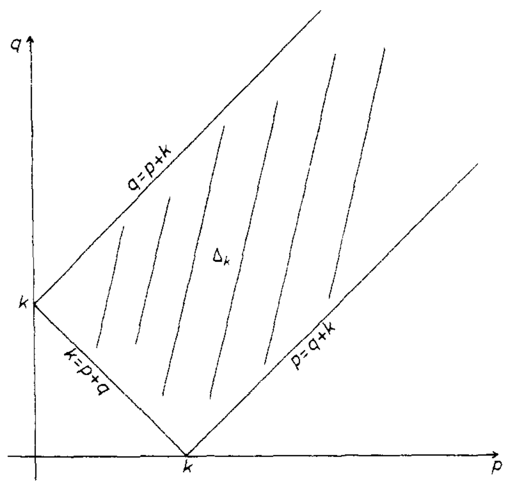


Figure 1. The domain Δ_k is limited by the triangular inequality $|p-q|\leq k\leq p+q$; k, p, q are magnitudes of vectors.

Because of the simplifications due to the assumptions of homogeneity and isotropy, all the tensor indices have disappeared, and the analytic continuation in d is straightforward.

An asymptotic expansion has been carried out on (3.5) as $d\to\infty$. It is seen that all the energy transfer comes from triads k, p, q with one right angle: in the emission term (the $E(p)E(q)$ term) because of the presence of $\sin^{d-3}\alpha$ most of the contribution comes from $\alpha\approx\pi/2$, i.e. $\boldsymbol{p}$ and $\boldsymbol{q}$ almost orthogonal, whereas in the absorption term (the $E(q)E(k)$ term) it comes from $\beta\approx\pi/2$.

The saddle point integration method reduces the integral over the Δ_k strip to line integrals shown in figure 2 and yields a factor $1/\sqrt{d}$. The $b^{(d)}$ coefficient is

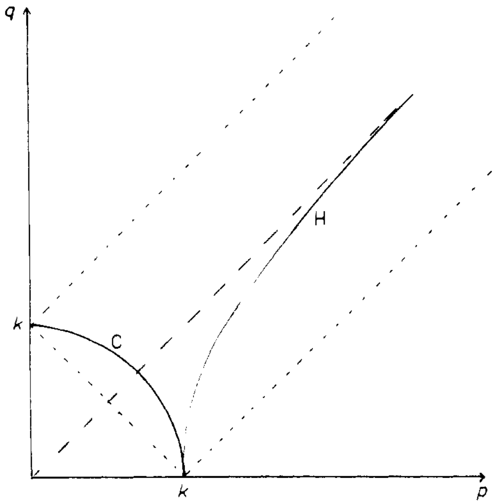


Figure 2. In the limit $d \to \infty$, all the energy transfer comes from triads k, p, q with one right angle. In the p, q plane, this yields a saddle-point integration over two lines, the 'emission circle' (C) and the 'absorption hyperbola' (H).

proportional to d and

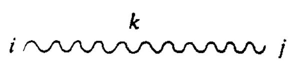
$$\frac{S_{d-1}}{(d-1)^2 S_d} \approx \frac{1}{d^2}\left(\frac{d}{2\pi}\right)^{1/2}.$$

Finally there remains a factor of the order of $1/d$ in front of the integrals which can be absorbed into a rescaled time $\tilde{t} = t/\sqrt{d}$ yielding the infinite-dimensional second-order expansion ($\nu = 0$)

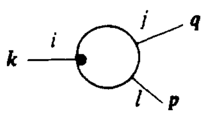
$$E(k;\tilde{t},\tilde{t}) = E(k;0,0) + 2\tilde{t}^2 k^3 \int_0^{\pi/2} \mathrm{d}\phi \Big(\cos^2\phi\, E(k\cos\phi;0,0) E(k\sin\phi;0,0) - \frac{1}{\cos^2\phi} E(k\tan\phi;0,0) E(k;0,0)\Big). \tag{3.6}$$

4. Higher-order expansions

In order to efficiently represent the higher-order terms in the expansion (2.4), diagrams will be used where



stands for $U_{ij}(\boldsymbol{k};0,0)$,

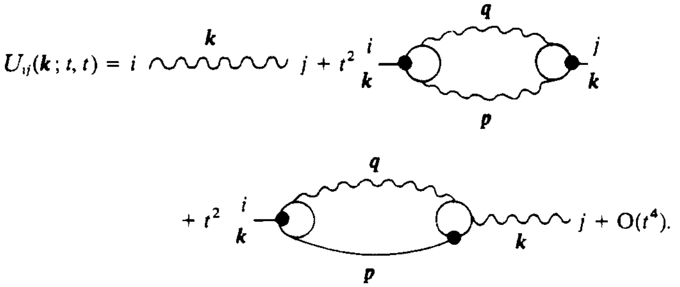


for the vertex $-\frac{1}{2}\mathrm{i}P_{ijl}(\boldsymbol{k})$, and

for 1

(the appearance of two kinds of lines, one for the covariance and one for the Green function, which assumes the value 1 for the inviscid linear case, is described in § 5).

To order t^2 the one-time spectrum is given by (multiplicities and viscous terms deleted)

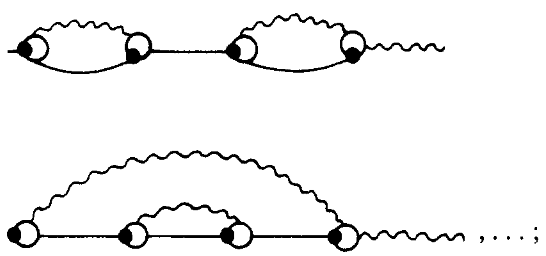


(4.1)

To order t^4, the following two classes of graphs appear, according to the structure of their Fourier integrals.

(i) 'Iterated' second-order graphs (in the sense that the integrals are essentially simple combinations of those present in (4.1)) such as

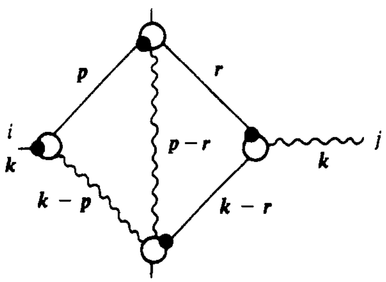
and



(4.2)

these give contributions $O((t/\sqrt{d})^4)$.

(ii) 'Vertex'-type graphs, such as



(4.3)

Such graphs can be evaluated using generalised spherical coordinates involving three angles, say

$$\theta_1 = (\boldsymbol{p}-\boldsymbol{r}, \boldsymbol{k}); \qquad \theta_2 = (\boldsymbol{k}-\boldsymbol{p}, \boldsymbol{k}); \qquad \theta_3 = (P_{\boldsymbol{k}}(\boldsymbol{p}-\boldsymbol{r}), P_{\boldsymbol{k}}(\boldsymbol{k}-\boldsymbol{p}))$$

where $P_{\boldsymbol{k}}$ denotes the projection onto the plane perpendicular to $\boldsymbol{k}$. The volume element will then contain high powers ($\sim d$) of the product $\sin\theta_1 \sin\theta_2 \sin\theta_3$. By saddle-point integration, for $d \to \infty$, this gives three orthogonality conditions ($\theta_1 \approx \theta_2 \approx \theta_3 \approx \pi/2$) and three factors $1/\sqrt{d}$. This is in contrast with the iterated second-order graphs which, because of their simple topological structure, give only two orthogonality conditions. At first sight this seems to imply that fourth-order vertex terms become negligible for $d \to \infty$. In fact, fourth-order vertex terms are as big as iterated terms, i.e. $O((t/\sqrt{d})^4)$, the reason being that the volume element in spherical coordinates with three angles is proportional to $S_{d-1}S_{d-2}$ (instead of $S_{d-1}S_{d-1}$ for iterated second order). Since $S_{d-2} \sim S_{d-1}\sqrt{d}$ this cancels the extra $1/\sqrt{d}$ factor.

This analysis has been extended to graphs of order $2n$, which are all found to be $O((t/\sqrt{d})^{2n})$. We therefore conclude that as a function of the rescaled time variable $\tilde{t} = t/\sqrt{d}$, all the terms in the formal Taylor expansion of $E(k;\, t, t)$ have a limit as $d \to \infty$. In addition, no class of diagrams becomes negligible in this limit. This is in contrast with the Heisenberg magnet in equilibrium critical phenomena, which has a four-point vertex and where a crucial role is played by the concept of diagrammatic loops: e.g. each loop gives a factor n (the number of spin components). In the limit of $n \to \infty$, these factors select from all possible diagrams a summable subset which then leads to the $1/n$ expansion. For the Navier–Stokes equation which has a three-point vertex, and is treated as an initial value problem, it can be shown that there does not exist any parallel to the concept of loop, and hence no direct counterpart to the $1/n$ expansion. Though one does not have the freedom to vary n (now the number of velocity components) and d independently in the Navier–Stokes equation because the number of velocity components and the number of space dimensions are by necessity equal, it is possible to introduce an additional integer parameter, m, by changing each component of the velocity into an $m \times m$ matrix. The possible consequences have not yet been explored.

5. Renormalised expansions

The formal Taylor series is a functional of the initial covariance $U_{ij}(\boldsymbol{k};\, 0, 0)$, or equivalently of what the covariance would be in the absence of interaction between the Fourier modes. Similarly, in the Reynolds number expansion, there appears the covariance as it evolves under the effect of only the linear viscous term in the Navier–Stokes equation, as well as the velocity amplitude Green function of this linear system. It is possible to revert these expansions and express the linear system's covariance and Green function in terms of those of the Navier–Stokes equation, and then substitute for the former in the Taylor or Reynolds-number expansions. This procedure is known as *renormalisation*. The perturbation theory obtained, whose lowest-order truncation is the DIA, is known as the renormalised expansion (Martin *et al* 1973, Kraichnan 1977, where vertex renormalisation is also discussed). This expansion also has a diagrammatic representation, a consolidated version of the Taylor series diagrams, where ∿∿∿∿ and ——— stand for the covariance and the mean velocity amplitude linear response function (Green function) of the Navier–Stokes equation respectively.

For example, let us consider the d-dimensional DIA equations, which read ($\nu = 0$):

$$\frac{\partial}{\partial t}E(k;\, t, t') = \frac{4S_{d-1}}{(d-1)^2 S_d}\int \mathrm{d}\tau \int_{\Delta_k} \mathrm{d}p\, \mathrm{d}q \frac{k}{pq} b^{(d)}(k, p, q)$$
$$\times(\sin^{d-3}\alpha\, k^2 E(p;\, t, \tau)E(q;\, t, \tau)G(k;\, t', \tau)$$
$$-\sin^{d-3}\beta\, p^2 E(q;\, t, \tau)E(k;\, \tau, t')G(p;\, t, \tau)) \tag{5.1}$$

$$\frac{\partial}{\partial t}G(k;\,t,t')=\delta(t-t')-\frac{4S_{d-1}}{(d-1)^2S_d}\int \mathrm{d}\tau\int_{\Delta_k}\mathrm{d}p\,\mathrm{d}q\,\frac{k}{pq}b^{(d)}(k,p,q)$$
$$\times\sin^{d-3}\beta\,p^2E(q;\,t,\tau)G(k;\,\tau,t')G(p;\,t,\tau). \tag{5.2}$$

$G(k;\,t,t')$ is related to the linear response function as usual:

$$G_{ij}(\boldsymbol{k};\,t,t')=P_{ij}(\boldsymbol{k})G(k;\,t,t'); \qquad G(k;\,t',t')=1.$$

Note the algebraic similarity to the second-order Taylor expansion. This similarity also holds for their respective diagrammatic representations. If the DIA itself is expanded in a Taylor series, the $O(t^2)$ terms coincide with those given by (4.1) and the $O(t^4)$ terms coincide with the iterated second-order diagrams (4.2).

The $d\to\infty$ limit of the DIA can be taken, provided the same time rescaling is performed, i.e. $\tilde{t}=t/\sqrt{d}$, with the result

$$\frac{\partial}{\partial \tilde{t}}E(k,\tilde{t},\tilde{t}')=2k^3\int \mathrm{d}\tilde{\tau}\int_0^{\pi/2}\mathrm{d}\phi\Big(\cos^2\phi\,E(k\cos\phi;\,\tilde{t},\tilde{\tau})E(k\sin\phi;\,\tilde{t},\tilde{\tau})G(k;\,\tilde{t}',\tilde{\tau})$$
$$-\frac{1}{\cos^2\phi}E(k\tan\phi;\,\tilde{t},\tilde{\tau})E(k;\,\tilde{\tau},\tilde{t}')G(k/\cos\phi;\,\tilde{t},\tilde{\tau})\Big) \tag{5.3}$$

$$\frac{\partial}{\partial \tilde{t}}G(k;\,\tilde{t},\tilde{t}')=\delta(\tilde{t}-\tilde{t}')$$
$$-2k^3\int \mathrm{d}\tilde{\tau}\int_0^{\pi/2}\frac{\mathrm{d}\phi}{\cos^2\phi}E(k\tan\phi;\,\tilde{t},\tilde{\tau})G(k;\,\tilde{\tau},\tilde{t}')G(k/\cos\phi;\,\tilde{t},\tilde{\tau}). \tag{5.4}$$

As is the case for $d=3$ (Kraichnan 1959) there exist inertial range solutions of the form (d large, ϵ = energy injection rate per unit mass and per unit primitive time, V_0 = root mean square velocity)

$$E(k)=Cd^{1/4}(\epsilon V_0)^{1/2}k^{-3/2}, \tag{5.5}$$

which suffer from the well known defect (Kraichnan 1964) of having an explicit dependence upon V_0, and hence being non-invariant under a random Galilean transformation: i.e. if the initial conditions are changed by adding energy at zero wavenumber,

$$E(k;\,0,0)\to E(k;\,0,0)+\tfrac{1}{2}\delta V_0^2\delta(k) \qquad V_0^2\to V_0^2+\delta V_0^2, \tag{5.6}$$

then because of the Galilean invariance of the Navier–Stokes equation

$$E(k;\,t,t)\to E(k;\,t,t)+\tfrac{1}{2}\delta V_0^2\delta(k), \tag{5.7}$$

which is inconsistent with (5.5). This defect can be cured by the *ad hoc* elimination from the Navier–Stokes equation of the so called 'non-local' interactions which couple eddies of widely disparate size. If this is done, then the Kolmogorov 1941 spectrum obtains:

$$E(k)=C'd^{1/3}\epsilon^{2/3}k^{-5/3}. \tag{5.8}$$

The characteristic time for the dynamics of the energy containing scales (presumably the time for the appearance of a singularity at zero viscosity) goes like $d^{1/2}$.

The renormalised approximations which retain terms in addition to those of the DIA, are called the vertex corrected approximations. Their diagrammatic representation contains, to the next order beyond the DIA, graphs which formally coincide with those in (4.3); their dependence upon d in the limit $d \to \infty$ is similar, i.e. the vertex corrections are not negligible compared to the DIA terms. Any other conclusion would lead to a serious difficulty because of the non-invariance of the DIA under a random Galilean transformation.

Since the renormalised expansion, in contrast to the Taylor expansion, is not *a priori* limited to short times, and since it is believed that $E(k; t, t)$ will assume a power law behaviour in the limit of large k after a finite time for $d > 2$ (even if the initial spectrum was confined to a finite wavenumber domain (Frisch *et al* 1976)) one is led to ask if the integrals contained in these expansions are convergent. They must be convergent in order that their dependence upon d may be simply deduced as was done above. Of course, this question is absent for the Navier–Stokes equation with cut-off non-local interactions. In the formal Taylor expansion of $E(k; t, t)$, assuming an initial power-law spectrum $E(k; 0, 0) \sim k^{-m}$ $(1 < m < 3)$ there are a number of superficial divergences (both infrared and ultraviolet) which disappear upon combining various terms or performing angular integrations. Some of these superficial divergences would probably never occur if we had used a Lagrangian expansion (Kraichnan 1977 and private communication). We have not yet investigated possible divergences in higher-order terms or moments. However, it has been shown rigorously for the Burgers equation ($\nu = 0$) in the context of the Taylor expansion, that there are no true divergences at any order for the energy spectrum and to order t^3 for the triple moment (Fournier and Frisch 1977b).

6. Pressure effects in infinite dimensions

In large space dimensions the incompressibility condition

$$\sum_{j=1}^{d} k_j V_j(\boldsymbol{k}) = 0$$

seems to impose only a rather weak constraint. Could it be that incompressibility and hence the pressure term become unimportant as $d \to \infty$? This question can be investigated by comparing two calculations. The first one with the Navier–Stokes vertex (equation (3.1)) involving the projection operator P_{ij} and the second one with a Burgers-like vertex having only the Kronecker δ_{ij}. To allow for compressibility we have to take a more general form of the covariance, namely

$$U_{ij}(\boldsymbol{k}) = \frac{2}{S_d k^{d-1}} \left(\frac{P_{ij}(\boldsymbol{k})}{d-1} E^{\mathrm{s}}(k) + \Pi_{ij}(\boldsymbol{k}) E^{\mathrm{c}}(k) \right)$$

where

$$\Pi_{ij}(\boldsymbol{k}) = k_i k_j / k^2.$$

E^{s} (s for solenoidal) is the incompressible spectrum and E^{c} the compressible spectrum. Starting with incompressible initial conditions we have shown that to order t^2 the pressure term makes no contribution to the incompressible part; but if pressure is deleted, a compressible part is generated in the t^2 coefficient which carries an energy $O(d^{-1})$ smaller than the compressible part. Nevertheless, by going to higher order (e.g. t^4) this incompressible part induces an O(1) change on the incompressible part. This clearly indicates that the two problems are not equivalent (at least within the framework of short-time expansions).

7. Summary and discussion

We restate now the principle technical results of the paper and discuss their physical content. We begin with three kinematic results. First it was pointed out to us by S Corrsin (1977, private communication) that for $d \to \infty$ the longitudinal and transverse correlation functions become identical. This is easily checked by expressing the incompressibility condition as is done in Batchelor (1953). Second, we have found in §§ 3 and 4 that for $d \to \infty$ non-linear interactions are confined to triads k, p, q having one right angle ($k^2 = p^2 + q^2$, or permutations). This is simply understood by noting that two independent isotropic unit vectors are almost surely orthogonal in infinite dimension (the variance of their scalar product is $1/d$). This has an important consequence for transfer: if initially the energy is confined to a wavenumber band $a < k < b$ then it can never be transferred to wavenumbers less than a since the interaction of two orthogonal wavevectors within that band necessarily results in a wavenumber larger than a (this can also be seen on the emission term of the infinite-dimensional DIA equation (5.3)). Third, from the probabilistic viewpoint, having infinite dimensionality introduces a very interesting new element into the problem: in finite-dimensional homogeneous isotropic turbulence, ensemble averages can be replaced by translational averages (if ergodicity holds); in infinite-dimensional turbulence it is likely that one can in addition take rotational averages or simply average over the group of cyclic permutations of the coordinates. For example, if we know the velocity vector at (almost) any particular point we can calculate the mean square energy per component as

$$\langle V^2 \rangle = \lim_{d\to\infty} \frac{1}{d} \sum_{i=1}^{d} V_i^2.$$

Turning to more dynamical questions we first stress that it has been found that to each order in perturbation the energy spectrum per mass has a *finite limit* provided a rescaled time variable, $\tilde{t} = t/\sqrt{d}$, is used. We can therefore truly speak of an infinite-dimensional turbulence problem. Now, it is easily checked, say on (2.4), that it is equivalent to rescale the time variable by $\sqrt{d}$ or to rescale the energy spectrum by d; the latter is equivalent to assuming finite energy *per component* rather than a finite total energy. It is of interest to recast this result entirely in physical space. As may be checked by writing the Fourier transformation for a function which depends only on the modulus of the d-dimensional position vector, the assumption that wavenumbers remain finite (implicitly made since they are not rescaled) implies that the typical Euclidean separation, r, of two points within an eddy is $O(\sqrt{d})$ whereas individual components of $\boldsymbol{r}$ are O(1). This, together with the assumption of finite energy per component imply that typical components of the strain tensor are $O(1/\sqrt{d})$ and that its typical eigenvalues are O(1)†. It is then not surprising to find that the characteristic dynamical time also becomes O(1).

† The order of magnitude of the strain tensor components can again be calculated by Fourier transformation and the orders of magnitude of the eigenvalues follow from a well known theorem on large random symmetric matrices (Wigner 1955, Edwards and Jones 1976).

We discuss now *pressure effects*. It might be asked if the incompressibility constraint $\boldsymbol{k} \cdot \boldsymbol{V}_k = 0$ still plays a significant role as $d \to \infty$. It has been found that it does indeed: as stated in § 6 it is not legitimate to replace the Navier–Stokes vertex (with incompressibility projection operators $P_{ij}(k)$) by a 'Burger's' vertex with δ_{ij} instead of P_{ij}; the use of a Burger's vertex would result in a significantly different DIA equation (with energy conservation problems in addition). Nevertheless, we notice that the pressure is given by a Poisson equation with a source

$$-\sum_{i,j} \left(\frac{\partial V_i}{\partial X_j}\right)\left(\frac{\partial V_j}{\partial X_i}\right)$$

which is a sum of a large number (d^2) of terms; hence the pressure and the pressure gradient should depend only weakly on individual components of the velocity and/or the velocity gradient. It may therefore be conjectured that if there is an anisotropy affecting a finite number of velocity components the pressure will not be able to restore isotropy over a time of the order of the dynamical time as is the case in three dimensions (Herring 1974, Herring and Schumann 1976, Rotta 1951, 1962).

Finally we want to comment on the non-vanishing of vertex corrections (§ 5) and the possible relevance to the question of intermittency. It is clear that in an expansion based on Eulerian coordinates, vertex corrections cannot drop out since the DIA (even in the limit of infinite dimensions) is not invariant under random Galilean transformations as are the primitive equations. Recently Kraichnan (1977) has introduced a mixed Eulerian–Lagrangian systematic expansion which is to each order invariant under random Galilean transformations and reproduces to lowest order the Lagrangian history direct interaction (which is known to produce the $k^{-5/3}$ spectrum; Kraichnan 1965). Such Galilean invariant renormalised expansions are particularly cumbersome beyond their lowest order and we have not tried to analyse their high-d limit. It might be tempting to investigate the question of intermittency in the sense of deviations to the Kolmogorov 1941 theory using such expansions but it must be stressed that the presence of vertex corrections, even in a Galilean invariant expansion, does not rule out the absence of intermittency. As pointed out to us by R H Kraichnan (1977, private communication) it could just mean that there are corrections to the Kolmogorov constant (the numerical constant in front of $\epsilon^{2/3}k^{-5/3}$) without corrections to the exponent.

At this point we would like to comment about a result on infinite-dimensional turbulence obtained by Kraichnan (1974b) in a rather different context. Assuming a prescribed homogeneous isotropic incompressible velocity field, with white-noise time dependence, Kraichnan shows that the distribution of the modulus of the gradient of a passive scalar is log normal and goes over into a deterministic distribution as $d \to \infty$ so that intermittency disappears. It might be argued that stretching of vortex lines (or whatever their d-dimensional substitute is) has some resemblance to the stretching of a passive vector and, hence, that the intermittency of the vorticity field might disappear as $d \to \infty$. However in Kraichnan's calculation the successive stretchings are totally uncorrelated so that the scalar gradient has the possibility of exploring all directions whereas in the Navier–Stokes equation the velocity or the velocity gradient is, loosely speaking, self-stretching and there is no reason to expect abrupt changes in the direction of stretching.

We believe that it is of major importance to find out what happens to intermittency as $d \to \infty$. Finite-dimensional turbulence might become calculable by $1/d$ expansion if the intermittency disappears or if it takes some extreme, calculable, form as it does in Burger's equation.

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References

Batchelor G K 1953 *The Theory of Homogeneous Turbulence* (Cambridge: Cambridge University Press)
Brissaud A, Frisch U, Leorat J, Lesieur M, Mazure A, Pouquet A, Sadourny R and Sulem P L 1973 *Ann. Geophys., Paris* **29** 539
Edwards S F and Jones R C 1976 *J. Phys. A: Math. Gen.* **9** 1595
Forster D, Nelson D R and Stephen M J 1976 *Phys. Rev. Lett.* **36** 867
—— 1977 *Phys. Rev.* A **16** 732
Fournier J-D and Frisch U 1977a *Phys. Rev.* A in the press
—— 1977b *Observatoire de Nice Preprint*
Frisch U, Lesieur M and Sulem P-L 1976 *Phys. Rev. Lett.* **37** 895
Frisch U, Sulem P-L and Nelkin M 1977 *J. Fluid Mech.* submitted for publication
de Gennes P G 1975 *Fluctuation, Instability and Phase Transition: Proc. NATO Adv. Study Inst., Jeilo, Norway Ser. B* ed. T Riste (Leiden: Noordhoff) p 1
Gerber P R and Fischer M E 1974 *Phys. Rev.* B **10** 4697
—— 1975 *J. Chem. Phys.* **63** 4941
Herring J R 1974 *Phys. Fluids* **17** 859 (1976 erratum **19** 117)
Herring J R and Schumann U 1976 *J. Fluid Mech.* **76** 755
Kolmogorov A N 1941 *C.R. Acad. Sci. URSS* **30** 301
—— 1962 *J. Fluid Mech.* **12** 82
—— 1968 *Sov. Phys.–Usp.* **10** 734
Kraichnan R H 1959 *J. Fluid Mech.* **5** 497
—— 1964 *Phys. Fluids* **7** 1723
—— 1965 *Phys. Fluids* **8** 575 (1966 erratum **9** 1884)
—— 1966 *Dynamics of Fluids and Plasmas* ed. S I Pai, (New York: Academic) p 239
—— 1970 *J. Fluid Mech.* **41** 189
—— 1974a *J. Fluid Mech.* **62** 305
—— 1974b *J. Fluid Mech.* **64** 737
—— 1977 *J. Fluid Mech.* **81** 385
Leibbrandt G 1975 *Rev. Mod. Phys.* **47** 849
Ma S 1976 *Modern Theory of Critical Phenomena: Frontiers in Physics Ser. No. 46* (Reading, Mass.: Benjamin)
Martin P C, Siggia E D and Rose H A 1973 *Phys. Rev.* A **8** 423
Nelkin M 1974 *Phys. Rev.* A **9** 388
—— 1975 *Phys. Rev.* A **11** 1737
Orszag S A 1976 *Statistical Theory of Turbulence: Les Houches Summer School of Physics 1973* (New York: Gordon and Breach) pp 235–374
Rose H A and Sulem P L 1977 *J. Physique* submitted for publication
Rotta J C 1951 *Z. Phys.* **129** 547
—— 1962 *Progress in Aeronautical Sciences* vol. 2 eds A Ferri, D Kucheman and L H G Stern (New York: Pergamon) p 1
Wigner E P 1955 *Ann. Math., Lpz.* **62** 548